Differentiation Problem Solving Answers

① Parabolic shaped hill: Turning point form:

$$y = a(x-b)^2 + c$$

Turning point: $(100, 50)$

$$y = a(x-100)^2 + 50$$

Find a by substituting $y=0$ and $x=0$

$$0 = a(0-100)^2 + 50$$

$$0 = 10000a + 50$$

$$\frac{-50}{10000} = a$$

$$-0.005 = a$$

Equation of hill: $y = -0.005(x-100)^2 + 50$

Differentiate y using Chain rule.

$$y = -0.005u^2 + 50 \quad \text{where} \quad u = x - 100$$

$$\frac{dy}{du} = -0.01u \qquad \frac{du}{dx} = 1$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = -0.01u \times 1$$

$$\frac{dy}{dx} = -0.01(x-100)$$

Substitute in $\frac{dy}{dx} = 0.4$

$$0.4 = -0.01(x-100)$$

$$= -0.01x + 1$$

$$\frac{0.4-1}{-0.01} = x = 60$$

Answer: When $x=60$ Ben can start riding his bike uphill.

(2) $y = x^2 + 7x + 12$

$\frac{dy}{dx} = 2x + 7$

Gradient of tangent at $x = -1$

$\frac{dy}{dx} = 2 \times -1 + 7 = -2 + 7 = 5$

Gradient of normal at $x = -1$

$m = \frac{-1}{5}$

Equation of normal: $y = mx + c$

$y = \frac{-1}{5}x + c$

when $x = -1$, $y = (-1)^2 + 7(-1) + 12$

$= 1 + -7 + 12$

$= 6$

sub $x = -1$, $y = 6$ into $y = \frac{-1}{5}x + c$

$6 = \frac{-1}{5} \times -1 + c$

$6 - \frac{1}{5} = c$

$c = 5\frac{4}{5}$

Equation of normal: $y = \frac{-1}{5}x + 5\frac{4}{5}$

Where the normal cuts the curve:

Solve simultaneous equations:

$y = x^2 + 7x + 12$

$y = \frac{-1}{5}x + 5\frac{4}{5}$

$\frac{-1}{5}x + 5\frac{4}{5} = x^2 + 7x + 12$

$0 = x^2 + 7\frac{1}{5}x + 6.2$

$0 = x^2 + 7\frac{1}{5}x + 6.2$

Solve using quadratic formula

$x = -6.2$ or -1

The normal intersects the curve at

$x = -6.2$ and $x = -1$

When $x = -6.2$, $y = (-6.2)^2 + 7(-6.2) + 12$
$= 7.04$

Answer: The normal cuts the curve again at $(-6.2, 7.04)$

(3) Volume of sphere $= \frac{4}{3}\pi r^3$

Rate of change of volume with respect to radius

$$\frac{dv}{dr} = \frac{4}{3} \times 3 \times \pi r^2$$

$$= 4\pi r^2$$

Rate of change of volume with respect to time

$$\frac{dv}{dt} = -0.1$$

Rate of change of radius with respect to time

$$\frac{dr}{dt} = ?$$

Chain rule equation: $\frac{dv}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}$

$$-0.1 = 4\pi r^2 \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-0.1}{4\pi r^2}$$

Sub in $r = 1$

$$\frac{dr}{dt} = \frac{-0.1}{4 \times \pi \times 1^2} = -0.008 \text{ cm/min } (3 \text{ dp})$$

④ $f(x) = \frac{10}{x}$

Change $f(x)$ to y

$$y = \frac{10}{x}$$

Rearrange to make x the subject.

$$xy = 10$$

$$x = \frac{10}{y}$$

Change x to $f^{-1}(y)$

$$f^{-1}(y) = \frac{10}{y}$$

Therefore

$$f^{-1}(x) = \frac{10}{x}$$