# Linear transformations with Matrices lesson 5 - Equation of the image of a curve

Magic Monk Tutorials

## 1 Find the equation of the image of the curve $y = x^2 - 2x$ reflected in the $x$ axis and plot this in the x-y plane.

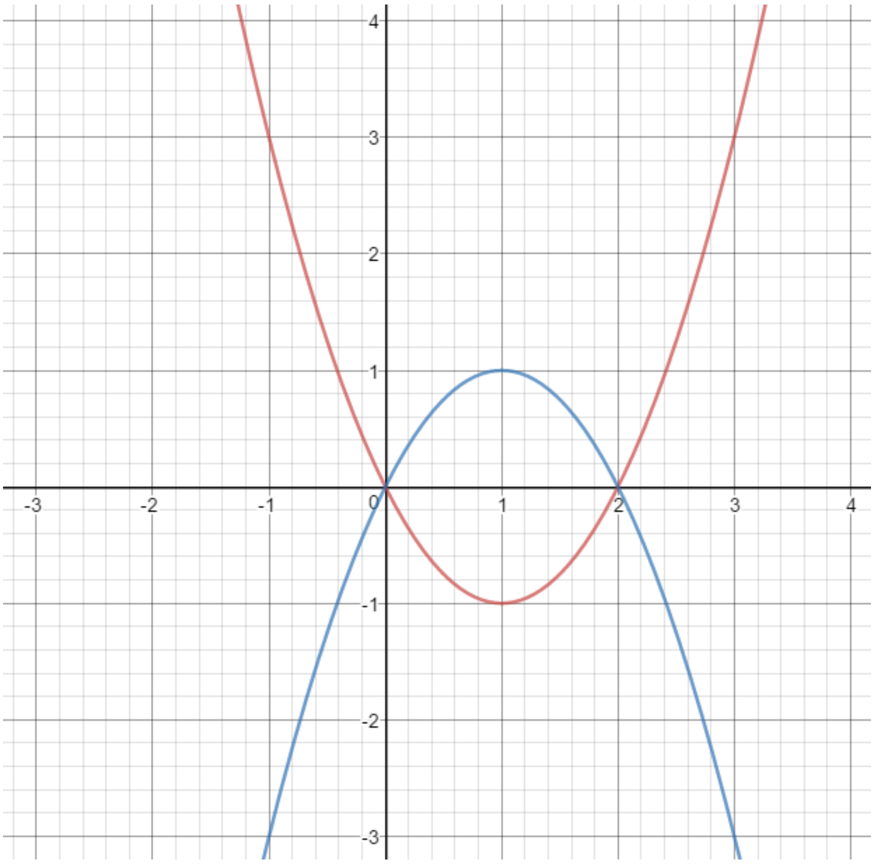
First, we seek the transformation matrix associated with reflection in the $x$ axis.
We note that reflection in the $x$ axis maps the points $(0,1) \mapsto (0,-1)$ and $(0,1) \mapsto (0,1)$.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now, by our definition of the transformation matrix,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Substituting $x = x'$ and $y' = -y$ into our equation, we get $-y' = x'^2 - 2x'$ and then $y' = 2x' - x'^2$. This transformed line is seen in blue below.

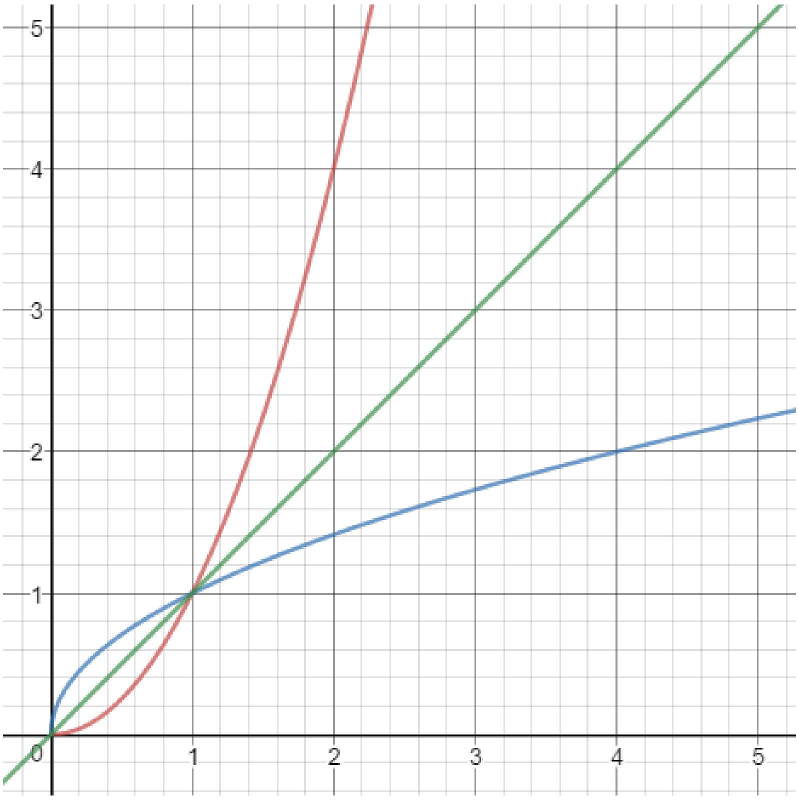


## 2 Find the equation of the line $y = x^2$ for $x \geq 0$ with the matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and plot the result in the x-y plane. Note that also plotted is the line $y = x$.

Apply our matrix T to some points $(x, y)$.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

Substitute $y = x'$ and $x = y'$ into our equation. $x' = y'^2$. Solving for y, $y' = \sqrt{x'}$. We do not need to worry about the $\pm$ since our line is only for $x \geq 0$. Our transformed curve is plotted in blue below. Note that we have reflected our curve about the line $y = x$, and that this resulted in a new function that is the inverse of the transformed function.




## 3 Bonus difficult question: Apply the shear transformation $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to the unit circle $y^2 + x^2 = 1$.

As before, use the matrix transformation formula.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + y \\ y \end{pmatrix}$$

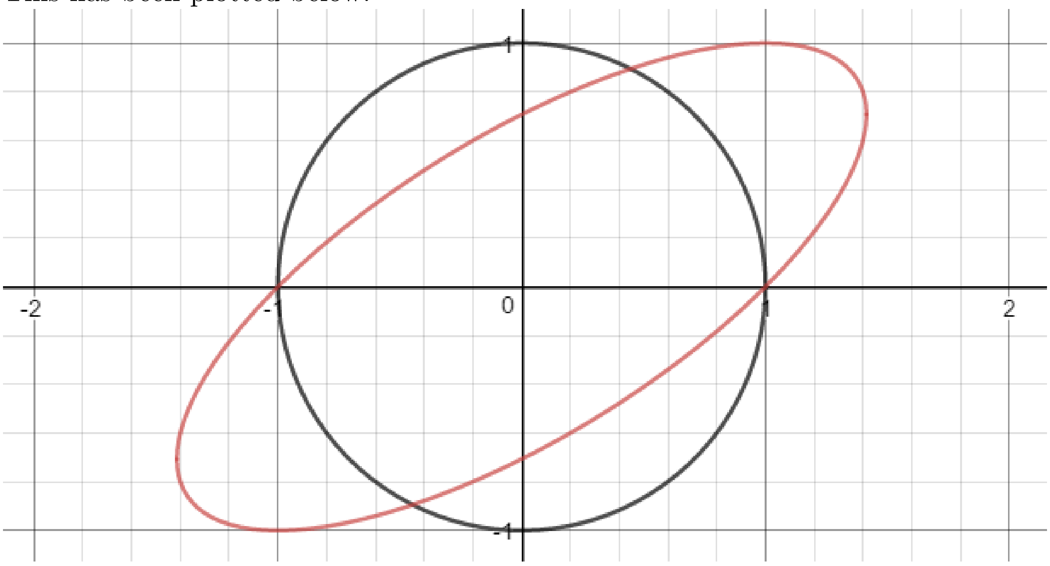
This results in the system of equations $x' = x + y$ and $y' = y$. In terms of $x$ and $y$, we have $y = y'$ and $x = x' - y'$. Substitute this into the formula for the unit circle.

$$y'^2 + (x' - y')^2 = 1$$
$$y'^2 + x'^2 - 2x'y' + y'^2 = 1$$
$$2y'^2 - 2x'y' + x'^2 - 1 = 0$$

We note that this is a quadratic in $y'$. This means we can solve for $y'$.

$$y' = \frac{2x' \pm \sqrt{4x'^2 - 8\left(x'^2 - 1\right)}}{4}$$
$$y' = \frac{2x' \pm \sqrt{8 - 4x'^2}}{4}$$
$$y' = \frac{x' \pm \sqrt{2 - x'^2}}{2}$$

This has been plotted below.




## 4 Bonus difficult question: Find the equation of the line $y = x^2 + x + 1$ for $x \geq 0$ with the matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The first part of the question has been done in question 2. We end up needing to solve for $y'$ in $0 = y'^2 + y' + 1 - x'$. Since this is a quadratic in $y'$, we may solve for $y'$.

$$y' = \frac{-1 \pm \sqrt{1 - 4\left(1 - x'\right)}}{2}$$
$$y' = \frac{-1 \pm \sqrt{x' - 3}}{2}$$