Linear transformations with Matrices lesson 7 - Image of a line rotated N degrees about the origin

Magic Monk Tutorials

1 Find the inverse of the Rotation matrix R_θ

$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \dfrac{1}{\cos\theta\cdot\cos\theta - \sin\theta\cdot -\sin\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \dfrac{1}{\cos(\theta)^2 + \sin(\theta)^2}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Using the fact that $\cos(\theta)^2 + \sin(\theta)^2 = 1$, we have

$R_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

It's worth noting that $R_\theta^{-1} = R_{-\theta}$, which is true since $R_{-\theta}$ is the clockwise rotation matrix.

2 Rotate the line $y = -x$ by $45°$ anticlockwise.

Find x and y in therms of y' and x'. Use the general transformation formula.

$\begin{pmatrix} x' \\ y' \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta x' + \sin\theta y' \\ -\sin\theta x' + \cos\theta y' \end{pmatrix}$

Substituting in $\theta = 45°$,

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos 45° x' + \sin 45° y' \\ -\sin 45° x' + \cos 45° y' \end{pmatrix} = \begin{pmatrix} x'/\sqrt{2} + y'/\sqrt{2} \\ -x'/\sqrt{2} + y'/\sqrt{2} \end{pmatrix}$

Substitute this into our formula for $y = -x$.

$-x'/\sqrt{2} + y'/\sqrt{2} = -x'/\sqrt{2} - y'/\sqrt{2}$

$2y'/\sqrt{2} = 0$ and therefore $y' = 0$ is our resulting line. This makes sense since $y = -x$ is the line $y = 0$ rotated by $45°$ clockwise.

3 Rotate the line $y = mx + c$ by $\theta°$.

Find x and y in therms of y' and x'. Use the general transformation formula as before.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = R_\theta^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta x' + \sin\theta y' \\ -\sin\theta x' + \cos\theta y' \end{pmatrix}$$

From this we have $x = \cos\theta x' + \sin\theta y'$ and $y = -\sin\theta x' + \cos\theta y'$. Substitute this into the equation $y = mx + c$ and rearrange for y'.

$$-\sin\theta x' + \cos\theta y' = m\left(\cos\theta x' + \sin\theta y'\right) + c$$
$$-\sin\theta x' + \cos\theta y' = m\cos\theta x' + m\sin\theta y' + c$$
$$\cos\theta y' - m\sin\theta y' = \sin\theta x' + m\cos\theta x' + c$$
$$y'\left(\cos\theta - m\sin\theta\right) = x'\left(\sin\theta + m\cos\theta\right) + c$$
$$y' = x'\frac{\sin\theta + m\cos\theta}{\cos\theta - m\sin\theta} + \frac{c}{\cos\theta - m\sin\theta}$$

We now have a general formula for rotating any line by any number of degrees.